Time Invariance for an Integrator

Prof. Brian L. Evans and Mr. Jaehong Moon
Dept. of Electrical and Computer Engineering
The University of Texas at Austin

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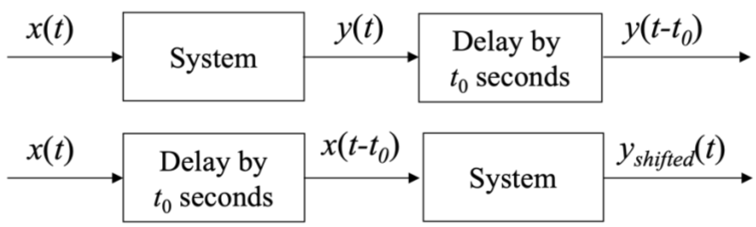
When evaluating system properties, we treat a system as a closed box and analyze the relationships between input signals and their corresponding output signals. This process assumes that after inputting a signal, we can return the system to its original state.

A continuous-time system with input signal $x(t)$ and output signal $y(t)$ is time-invariant (shift-invariant) if whenever the input signal is delayed by t_0 seconds, then the output signal will always be delayed by t_0 seconds as well for all real values of t_0.

A way to visualize the time-invariance property is to show the equivalence between



That is, does $y_{shifted}(t) = y(t - t_0)$ for all possible real constant values for t_0?

Integrator. An integrator captures the idea of conservation or storage [1]. "Capacitors can be modeled as integrators (capacitors are reservoirs for electric charge)." [1] The integrator integrates the input signal $x(t)$; i.e., the output of the integrator is

$$y(t) = \int_{-\infty}^{t} x(\lambda)\, d\lambda$$

Two-sided infinite observation. For this case, we observe the input and output signals for all time, i.e. $-\infty < t < \infty$. We can determine if the system is time-invariant as follows. We delay the input signal $x(t - t_0)$ and analyze the resulting output signal

$$y_{shifted}(t) = \int_{-\infty}^{t} x(\lambda - t_0)\, d\lambda$$

and see if it is equal to $y(t - t_0)$. We can use a change of variables $\xi = \lambda - t_0$ which means that $d\xi = d\lambda$, $\xi_{upper} = t - t_0$ and $\xi_{lower} = -\infty$ to obtain

$$y_{shifted}(t) = \int_{-\infty}^{t-t_0} x(\xi)\, d\xi = y(t - t_0)$$

Conclusion: An integrator under two-sided observation in time is time-invariant.

One-sided infinite observation. In this case, we observe the input and output signals of the integrator for $t \geq 0^-$. Time 0^- means a time of 0 seconds before occurrence of a Dirac delta occurring at the origin. We will only allow positive values of t_0 so that the delay for the input and/or output signals will be at times being observed.

Under what conditions will the integrator be time-invariant? [2]

The integrator output for $t \geq 0^-$ is

$$y(t) = \int_{-\infty}^{t} x(\lambda)\, d\lambda = \int_{-\infty}^{0^-} x(\lambda)\, d\lambda + \int_{0^-}^{t} x(\lambda)\, d\lambda = C_0 + \int_{0^-}^{t} x(\lambda)\, d\lambda$$

The scalar constant C_0 captures the result of the integration of the input signal over time not observed from $-\infty < t < 0$. Delaying the output signal by t_0 seconds gives

$$y(t - t_0) = C_0 + \int_{0^-}^{t-t_0} x(\lambda)\, d\lambda \qquad (1)$$

for $t \geq 0^-$. For $0^- < t < t_0$, the integral accesses values of the input signal $x(t)$ from $-t_0 < t < 0$ which we are not able to observe.

When delaying the input signal $x(t - t_0)$, the resulting output signal for $t \geq 0^-$ is

$$y_{shifted}(t) = C_0 + \int_{0^-}^{t} x(\lambda - t_0)\, d\lambda$$

That is, the integrator has its own time reference, and shifting the input or output signal in time does not affect the clock internal to the integrator.

To help compare $y(t - t_0)$ and $y_{shifted}(t)$, we perform algebraic manipulations on

$$y_{shifted}(t) = C_0 + \int_{0^-}^{t} x(\lambda - t_0)\, d\lambda$$

We use a substitution of variables $\xi = \lambda - t_0$ which means that $d\xi = d\lambda$, $\xi_{upper} = t - t_0$ and $\xi_{upper} = -t_0$ to obtain

$$y_{shifted}(t) = C_0 + \int_{-t_0}^{t-t_0} x(\xi)\, d\xi$$

$$y_{shifted}(t) = C_0 + \int_{-t_0}^{0^-} x(\xi)\, d\xi + \int_{0^-}^{t-t_0} x(\xi)\, d\xi$$

$$y_{shifted}(t) = C_0 + C_1(t_0) + \int_{0^-}^{t-t_0} x(\xi)\, d\xi \qquad (2)$$

For (1) and (2) to be equal for $t \geq t_0$, $C_1(t_0)$ has to be zero for all values of t_0. That is, when integrating $x(t)$ with respect to t from $-t_0$ to 0^-, the answer is always 0 regardless of the value of t_0. The only way for this to happen is if $x(t)$ has amplitude of zero for $-t_0 < t < 0^-$. This means that as $t_0 \to \infty$, $C_0 \to 0$.

Alternately, as $t_0 \to \infty$, $C_1(t_0) \to C_0$. In the limit, a necessary condition for (1) and (2) to be equal for $t \geq t_0$ is for $C_0 = 0$ to solve $C_0 = 2\, C_0$.

Conclusion: An integrator observed for $t \geq 0^-$ is time-invariant if the initial condition is 0.

References

[1] Pedro Albertos and Iven Mareels, *Feedback and Control for Everyone*, Springer, 2010.

[2] Stack Exchange, "Consider the integrator and check for time invariance", April 14, 2019.